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PAPERS

ELASTIC RESTRAINT EQUATIONS FOR SEMI-RIGID CONNECTIONS

By J. E. Lothers,[1] M. ASCE

Synopsis

The restraining effects of semi-rigid, girder-column connections have received considerable attention within recent years in both the United States and Great Britain. Excellent literature on the subject is now available, and methods of analyses which take into account the restraining effects of these connections have been proposed.[2,3,4] On the whole, these proposed methods would be conducive to economy in metal structures except for one fact. Unfortunately, expensive and time-consuming laboratory tests must be performed in advance, on any given semi-rigid connection in order to obtain a constant which measures the extent of its rigidity and which is necessary for analysis. Proposed equations for the computation of this constant, and a table of computed values for standard beam connections, are presented in this paper.

In addition, a proposed new method is advanced for the analysis and design of one type of steel bracket which is based on the bending strength of the connecting angles as opposed to the present method of basing its strength on the rivets. This proposed method was evolved as a by-product of the main problem of finding elastic restraint equations for semi-rigid connections.

The notation used in the paper is introduced directly in the text and presented alphabetically in the Appendix.

Introduction

The application of the methods of slope deflections and moment distribution to the analysis of rigid building frames is now accepted practice. In the case

Note.—Written comments are invited for publication; the last discussion should be submitted by July 1, 1950.

[1] Prof. of Architecture, Oklahoma Agri. and Mech. College, Stillwater, Okla.

[2] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 993.

[3] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *ibid.*, Vol. 101, 1936, p. 524.

[4] "First, Second, and Final Reports," Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, 1931–1936.

of continuous reinforced concrete frames with properly reinforced joints, this practice is justified. It is also justified in welded steel buildings where the ends of the beam flanges are welded directly to the column, provided the column flanges are reinforced against any strain.

On the other hand, where the girder-column connection is made through the medium of angles or tees, whether riveted or welded, the deflection due to bending in their outstanding legs gives rise to a rotation, ϕ, of the beam which is in addition to the rotation, θ, of the entire joint. This type of connection has become known among structural engineers as a semi-rigid connection. To apply one of the methods of so-called exact analysis to a semi-rigid connection, without some correction, results in an incorrect distribution of moment along the beam. To ignore the restraint provided, on the other hand, is wasteful. If a correction based on ϕ be incorporated in the usual methods of exact analysis, it is possible to determine a proper distribution of moment between the ends and the intermediate points of the beam and thus take intelligent advantage of the economy provided by these semi-rigid connections.

Literature on the subject[3] makes use of a connection constant, Z, such that,

$$Z = \frac{\phi}{M} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Bruce Johnston, M. ASCE, and Edward H. Mount[2] have defined Z (Eq. 1) as the angle change for unit moment. It is a factor which, when used in conjunction with the E and I of a member connected by the semi-rigid joint, becomes the necessary correction agent to render the usual exact methods of analysis applicable. The reciprocal, $1/Z$, is the slope of the moment-rotation curve[3] introduced by J. Charles Rathbun, M. ASCE. The symbol ϕ may be defined as the angle of strain in the semi-rigid connection. It is the number or radians by which the angle between the beam and the column differs from a right angle. For the purpose of this paper, M will be defined as the safe resisting moment of the connection (see Appendix).

Analysis

Steps of Procedure.—To follow the steps in the development of elastic restraint equations as applied to standard connections, it is necessary to visualize what happens to a pair of connection angles supporting the end of a loaded beam. A horizontal pull is exerted by the top of the beam which puts bending in the legs of the angles and pulls their heels away from the column. The amount of this deflection (Δ_b, Fig. 1) is a maximum at the top of the connection angles and zero at their neutral, bending axis, y inches below the top (see Fig. 2).

The procedure for arriving at an equation for Z, as applied to standard connections, may be divided into the following six steps:

1. Find an expression for the critical moment in the legs of the connection angles due to the outward pull of the top of the beam, and assuming no give in the rivets.[3,5,6]

[5] "Permissible Stresses on Rivets in Tension," by C. R. Young and W. B. Dunbar, *Bulletin No. 8*, School of Eng. Research, Univ. of Toronto, Toronto, Ont., Canada, 1928, p. 7.

[6] "Tension Tests of Rivets," by W. M. Wilson and W. A. Oliver, *Bulletin No. 210*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill., 1930.

2. Find an expression for the deflection, Δ_b, of the upper ends of the connection angles outward from the face of the column (Figs. 1 and 2).

3. Locate the neutral, bending axis of the connection as governed by the elastic behavior of the legs of the angles, and find an expression for its distance, y, down from the top.

4. The angle of strain, ϕ, equals the deflection, Δ_b, of the heels of the connection angles, divided by y; that is,

$$\phi = \frac{\Delta_b}{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

5. Find an expression for the safe resisting moment, M, of the connection, based on the elastic restraint of the legs of the angles.

6. Finally

$$Z = \frac{\phi}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

in which Z will be expressed in terms of the dimensions and known functions of the connection angles.

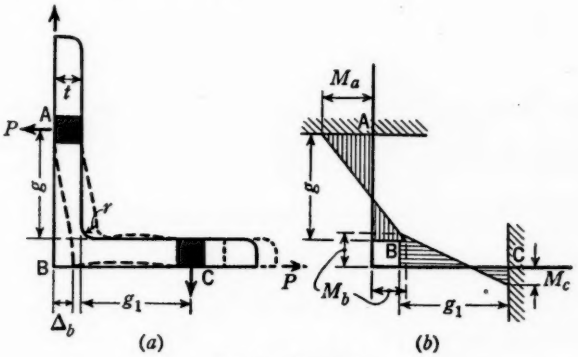

FIG. 1.—SYMBOLS

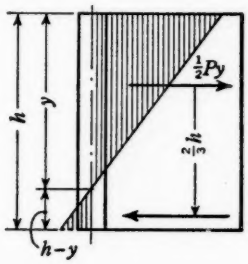

FIG. 2.—RESISTING COUPLE FOR EACH ANGLE

The Critical Moment in Connection Angles.—Expressions for the bending moments at points A, B, and C for a load, P, as shown in Fig. 1, may be derived by the method of slope deflections. The basic equations with known values substituted follow:

$$M_{ab} = \frac{2EI}{g(2\theta_a + \theta_b - 3r_{ab})} = \frac{2EI}{g(\theta_b - 3r_{ab})} \quad \ldots\ldots\ldots\ldots (3)$$

$$M_{ba} = \frac{2EI}{g(2\theta_b + \theta_a - 3r_{ab})} = \frac{2EI}{g(2\theta_b - 3r_{ab})} \quad \ldots\ldots\ldots\ldots (4)$$

$$M_{bc} = \frac{2EI}{g_1(2\theta_b + \theta_c - 3r_{bc})} = \frac{2EI}{g_1(2\theta_b)} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

and

$$M_{cb} = \frac{2EI}{g_1(2\theta_c + \theta_b - 3r_{bc})} = \frac{2EI}{g_1(\theta_b)} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Referring to Fig. 1, the following relations are evident:

$$M_{ba} + M_{bc} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and

$$M_{ab} + M_{ba} = P\,g \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

If the proper substitutions, Eq. 3 to Eq. 5, be made in Eqs. 7 and 8, and if the latter are solved simultaneously for θ_b and r_{ab}, the following expressions result:

$$\theta_b = -\frac{P\,g^2}{2\,E\,I} \times \frac{g_1}{g_1 + 4\,g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and

$$r_{ab} = -\frac{P\,g^2}{3\,E\,I} \times \frac{g + g_1}{4\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Finally, these values of θ_b and r_{ab} are substituted in the basic equations to derive M_a, M_b, and M_c:

$$M_a = P\,g \times \frac{2\,g + g_1}{4\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$M_b = P\,g \times \frac{2\,g}{4\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and

$$M_c = P\,g \times \frac{g}{4\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

The Deflection, Δ_b.—In Fig. 1(*b*), if the moment diagram be replaced by the $\frac{M}{E\,I}$-diagram, the deflection, Δ_b, of the heel of the angle outward from the support may be obtained by taking moments of that part of the $\frac{M}{E\,I}$-diagram on the leg AB about point B. This will give the deflection in a direction perpendicular to the tangent at point A. However, a simpler method of computing this deflection follows directly from Eq. 10. By definition, r_{ab} equals the deflection, Δ_b, divided by g. Substitute the value of r_{ab} from Eq. 10, ignore the minus sign, and solve for Δ_b—

$$\Delta_b = \frac{P\,g^3}{3\,E\,I} \times \frac{g + g_1}{4\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The Angle of Strain, ϕ.—As stated in step 4 (Eq. 2*a*), ϕ is equal to the deflection, Δ_b, outward from the heels of the angles, at the top of the connection, divided by y. Therefore,

$$\phi = \frac{\Delta_b}{y} = \frac{P\,g^3}{3\,E\,I\,y} \times \frac{g + g_1}{4\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

An expression for P may be found from Eq. 11. The resisting moment of a 1-in. strip at the top of the connection is governed by its strength at point A—that is, $M_a = \frac{s\,t^2}{6}$, in which s is the bending stress and t is the thickness of

the angle. If the value of M_a in Eq. 11 be substituted, the pull, P, on the top inch of each angle, is

$$P = \frac{s\,t^2}{6\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

In Eq. 15, substitute the value of P from Eq. 16, and for I substitute $t^3/12$:

$$\phi = \frac{2\,g^2\,s}{3\,E\,t\,y} \times \frac{g + g_1}{2\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The Resisting Moment, M, of the Connection.—Let P equal the maximum pull, in pounds per inch of depth, per angle, at the top of the connection. The average pull, per inch of depth for each angle, above the netural axis, is $\frac{1}{2}\,P\,y$. Taking moments about the lower force of the resisting couple in Fig. 2, the

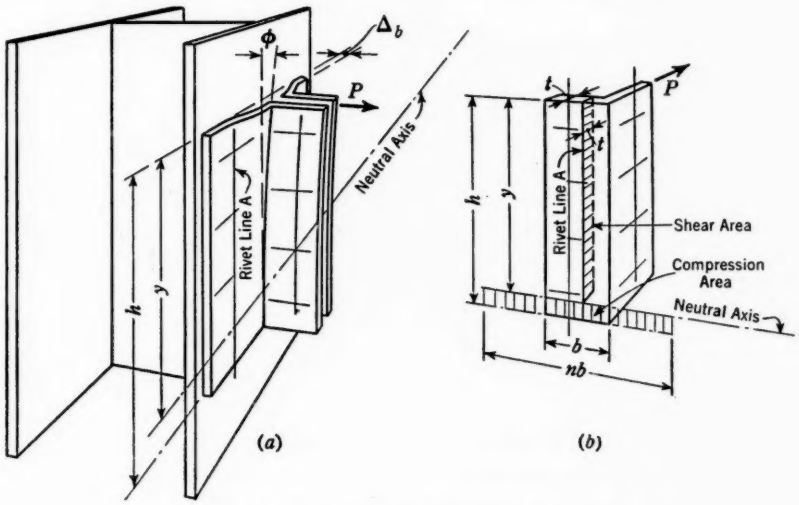

FIG. 3.—NEUTRAL AXIS AND TRANSFORMED SECTION

resisting moment is found to be $\frac{P\,y\,h}{3}$ for one angle, or $\frac{2\,P\,y\,h}{3}$ for the connection. The final expression for the total resisting moment, with the value of P substituted from Eq. 16, is

$$M = \frac{h\,s\,t^2\,y}{9\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

The Neutral Axis.—The problem of locating the neutral axis of the connection is analogous to that of the transformed section in reinforced concrete beam analysis. This follows from the fact that the compression area ($n\,b(h - y)$) below the neutral axis, as shown in Fig. 3(*b*), is opposed by a shear area ($t\,y$) above. The critical section is a right section of the column-connected leg of the angle along rivet line A at the top of the angle. This so-

called critical section is subjected to a bending stress which is difficult to associate directly with the compression area below the neutral axis. On the other hand, the shear area, indicated in Fig. 3(*b*), is directly opposed to the aforementioned compression area. Fortunately, the relation between the shearing stress on this area and the critical bending stress along rivet line A can be expressed. This relationship gives rise to a stress-ratio factor, n, which is analogous to the $n = E_s/E_c$ used in establishing the transformed section in reinforced concrete beam analysis.

Eq. 16 yields the value of the pull, P, on the top inch of one angle in terms of the bending stress, s, along rivet line A. The pull P is also the total shear on the top inch of the column-connected leg of each angle. Therefore, the unit shear, v, on the top inch is P/t. Substitute the value of P from Eq. 16:

$$v = \frac{s\,t\,(4\,g + g_1)}{6\,g\,(2\,g + g_1)} \qquad (19)$$

in which s is the bending stress in the column-connected leg of the angle.

Eq. 19 establishes the desired relationship between the shear area, $t\,y$, and the critical bending section along rivet line A and, accordingly, provides the key to the desired stress-ratio factor, n. It is assumed that the ratio of bending stress to strain in the column-connected leg of the angle along rivet line A is the same as that in compression below the neutral axis. Therefore, to arrive at the properly shaped transformed area in Fig. 3(*b*), either the shear area, $t\,y$, must be multiplied by the coefficient of s in Eq. 19, or the compression area, $b\,(h - y)$, below the neutral axis, must be divided by that coefficient. The latter method will be adopted and, accordingly,

$$n = \frac{6\,g\,(2\,g + g_1)}{t\,(4\,g + g_1)} \qquad (20)$$

Equating the statical moment of the shear area, $t\,y$, above the neutral axis (Fig. 3(*b*)) to that of the compression area, $n\,b\,(h - y)$, below, taken about the neutral axis, gives rise to a quadratic equation in y which (with the plus sign before the radical discarded) becomes

$$y = \frac{h\,(n\,b - \sqrt{n\,b\,t})}{n\,b - t} \qquad (21)$$

The Semi-Rigid Connection Factor, Z.—This factor has been defined previously as the angle of strain for unit moment for any given connection; that is, $Z = \dfrac{\text{angle of strain}}{\text{moment}} = \dfrac{\phi}{M}$. Substitute in Eq. 1 the values of ϕ and M from Eqs. 17 and 18, respectively, and simplify:

$$Z = \frac{6\,g^3}{E\,h\,t^3\,y^2} \times \frac{g + g_1}{4\,g + g_1} \qquad (22)$$

Strengthening Effect of the Fillet.—Inasmuch as g and g_1 are measured from the centers of their respective rivets to the faces of the adjacent legs of the

angle, the fillet functions as a haunch for both legs, and adds stiffness to the connection. To have taken the fillet into consideration in the derivation of Eqs. 18 and 22 for M and Z would have entailed the derivation of a different set of equations for each web thickness of beam and for each fillet radius involved. This follows from the fact that g changes in length with each thickness of web straddled, while the fillet remains constant for that particular connection. In addition, there are two different radii of fillet represented in the standard connections.

The effect of the fillet is not important so far as it affects Z. To include it would be inconsistent and misleading in the inferred accuracy to be obtained—not to mention the complications. There are factors that contribute to the rotation of the connection which have been neglected in the derivation of Eq. 22. Among them may be mentioned shear deflection in the column-connected leg of the angles, possible rivet slip, and strain in the column flanges. These should compensate for neglect of the fillet.

On the other hand, it is important that the true resisting moment of the connection be known for the purpose of applying the following proposed new method of steel bracket design.

The analysis leading to the derivation of Eq. 18 for the resisting moment of a connection treated the two legs of the angle as short, prismatic beams, fixed at their outer ends, A and C, and restrained at their common end, B. However, the fillet provides a comparatively long haunch at the B-end of both beams. To include the haunch in the slope deflection analysis it is necessary to rewrite Eqs. 3 to 6 in the following general forms:[7]

$$M_{ab} = \frac{E I_o}{g\,[C_1\,\theta_a + C_2\,\theta_b - (C_1 + C_2)\,r_{ab}]} \quad \ldots\ldots\ldots\ldots (23)$$

$$M_{ba} = \frac{E I_o}{g\,[C_2\,\theta_a + C_3\,\theta_b - (C_2 + C_3)\,r_{ab}]} \quad \ldots\ldots\ldots\ldots (24)$$

$$M_{bc} = \frac{E I_o}{g_1\,[C_4\,\theta_b + C_5\,\theta_c - (C_4 + C_5)\,r_{bc}]} \quad \ldots\ldots\ldots\ldots (25)$$

and

$$M_{cb} = \frac{E I_o}{g_1\,[C_5\,\theta_b + C_6\,\theta_c - (C_5 + C_6)\,r_{bc}]} \quad \ldots\ldots\ldots\ldots (26)$$

in which I_o, in this case, is the moment of inertia of a typical section taken along the prismatic parts of g or g_1. For a prismatic beam C_1 and C_2 are equal to 4 and 2, respectively. For a haunched beam the C-values must be computed.

As previously explained, a general equation analogous to Eq. 18, but with the fillet effects included, is impossible. By setting up equations for a typical connection, however, it is possible to establish a correction factor by the simple method of comparing the results obtained for that connection with and without fillet effects. The standard AISC (American Institute of Steel Construction)[8,9] K5 connection straddling a $\frac{3}{4}$-in. web will be chosen for this

[7] "Statically Indeterminate Structures," by L. C. Maugh, John Wiley & Sons, Inc., New York, N. Y., 1946, pp. 132–137.

[8] "Steel Construction," AISC Manual, New York, N. Y., 1947, pp. 252–261.

[9] *Ibid.*, p. 285.

purpose. The necessary data are as follows: $g = 1.9375$ in.; $g_1 = 1.8125$ in.; $t = \frac{7}{16}$ in.; and $r = \frac{1}{2}$ in. The C-values computed by the method of L. C. Maugh,[7] M. ASCE, are, for C_1 to C_6, respectively, 2.26, 1.46, 3.04, 3.37, 1.62, and 2.43. If each of the latter values be substituted for its respective C-values in Eqs. 23 through 26 and the same procedure followed as was employed with respect to Eqs. 3 through 6, the following equations for end moments may be obtained:

$$M_a = P\,g \times \frac{2.64\,g + g_1}{5.84\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

$$M_b = P\,g \times \frac{3.20\,g}{5.84\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

and

$$M_c = P\,g \times \frac{1.54\,g}{5.84 + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

Eqs. 27 through 29 provide for the strengthening effect of the fillet and are analogous to Eqs. 11 through 13. Proceeding in a manner similar to that followed in obtaining Eq. 18, an analogous expression for the resisting moment of the standard AISC connection[8,9] K5, and one which includes the effect of the fillet, may be found:

$$M = \frac{h\,s\,t^2\,y}{9\,g} \times \frac{5.84\,g + g_1}{2.64\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Comparing Eqs. 18 and 30 it may be seen that they differ only in their coefficients of g in the second fraction. Setting $g = g_1$, the second fraction is 5/3 in Eq. 18 and 6.84/3.64 in Eq. 30. The latter is 1.13 times the former. Therefore, the correction factor by which the right-hand side of Eq. 18 should be multiplied in order to take into account the strengthening effect of the fillet is 1.13. The resulting equation is very nearly—

$$M = \frac{h\,s\,t^2\,y}{8\,g} \times \frac{4\,g + g_1}{2\,g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Inasmuch as the constant, 1.13, was established by a comparison between the results from Eq. 18 and those taken from a fillet-based equation for one web thickness only, it might well be questioned that sufficient evidence has been presented to prove that the aforementioned constant applies throughout the entire range of web thicknesses. A set of fillet-based equations similar to Eqs. 27 through 30 was derived for each of the web thicknesses, $\frac{1}{2}$ in., $\frac{3}{4}$ in., and 1 in. Although the pair of constants, K_1 and K_2 in the $\frac{K_1\,g + g_1}{K_2\,g + g_1}$-term of the equation for M were different for all three of the web thicknesses considered, they did establish an important fact. It was found that, if g and g_1 each be set equal to unity in the foregoing term, it was constant for all three web thicknesses, as far as could be detected by slide rule, and, accordingly, bore a constant ratio to the analogous term in Eq. 18. It is believed, therefore, that the use of the 1.13-factor, as indicated, is justified.

TABLE 1.—Constants for Standard American Institute of Steel Construction Connections

Web thickness (in.)	Length, g (in.)	(*A*) Series A–B–H–K ($r_{A-B} = \frac{3}{8}$ in. and $r_{H-K} = \frac{1}{2}$ in.)			(*B*) Series HH–KK ($r = \frac{1}{2}$ in.)		
		Distance, y (in.)	Moment, M (in-lb)	$\frac{1}{Z \times 10^8}$	Distance, y (in.)	Moment, M (in-lb)	$\frac{1}{Z \times 10^8}$
(1)	(2)	(3)	(4)	(5)	(3)	(4)	(5)
(*a*) Connection 3: $h = 8.5$; $t = \frac{1}{4}$; and $g_1 = 1.8750$							
....	....	$y = 0.936\ h$			$y = 0.947\ h$		
$\frac{5}{8}$	2.0625	7.95	19,500	0.402	8.04	19,800	0.411
$\frac{1}{2}$	2.1250	7.96	19,000	0.371	8.05	19,300	0.379
$\frac{3}{8}$	2.1875	7.96	18,600	0.343	8.05	18,900	0.351
$\frac{1}{4}$	2.2500	7.97	18,100	0.319	8.06	18,400	0.326
(*b*) Connection 4: $h = 11.5$; $t = \frac{1}{4}$; and $g_1 = 1.8750$							
....	....	$y = 0.936\ h$			$y = 0.947\ h$		
$\frac{3}{4}$	2.0000	10.75	36,700	1.079	10.88	37,100	1.105
$\frac{5}{8}$	2.0625	10.75	35,800	0.994	10.88	36,100	1.018
$\frac{1}{2}$	2.1250	10.76	34,800	0.917	10.89	35,300	0.884
$\frac{3}{8}$	2.1875	10.77	34,300	0.850	10.90	34,500	0.870
$\frac{1}{4}$	2.2500	10.78	33,200	0.789	10.91	33,600	0.808
(*c*) Connection 5: $h = 14.5$; $t = \frac{1}{16}$; and $g_1 = 1.8125$							
....	....	$y = 0.924\ h$			$y = 0.938\ h$		
$\frac{3}{4}$	1.9375	13.39	80,900	3.692	13.58	82,000	3.798
$\frac{5}{8}$	2.0000	13.40	78,800	3.392	13.59	80,000	3.489
$\frac{1}{2}$	2.0625	13.41	76,800	3.126	13.60	77,900	3.215
$\frac{3}{8}$	2.1250	13.43	74,800	2.888	13.62	76,000	2.970
(*d*) Connection 6: $h = 17.5$; $t = \frac{1}{16}$; and $g_1 = 1.8125$							
....	....	$y = 0.924\ h$			$y = 0.938\ h$		
$\frac{3}{4}$	1.9375	16.16	117,900	6.490	16.39	119,500	6.676
$\frac{5}{8}$	2.0000	16.17	114,700	5.961	16.40	116,500	6.131
$\frac{1}{2}$	2.0625	16.19	111,800	5.499	16.41	113,500	5.650
$\frac{3}{8}$	2.1250	16.20	109,200	5.071	16.43	109,300	5.216
(*e*) Connection 7: $H = 20.5$; $t = \frac{1}{16}$; and $g_1 = 1.8125$							
....	....	$y = 0.924\ h$			$y = 0.938\ h$		
$\frac{3}{4}$	1.9375	18.93	161,700	10 433	19.20	164,100	10.733
$\frac{5}{8}$	2.0000	18.95	157,600	9.590	19.21	159,700	9.854
$\frac{1}{2}$	2.0625	18.96	152,600	8.835	19.23	155,700	9.088
(*f*) Connection 8: $h = 23.5$; $t = \frac{1}{16}$; and $g_1 = 1.8125$							
....	....	$y = 0.924\ h$			$y = 0.937\ h$		
$\frac{3}{4}$	1.9375	21.72	212,800	15.745	22.01	215,600	16.168
$\frac{5}{8}$	2.0000	21.73	207,200	14.455	22.02	209,900	14.843
$\frac{1}{2}$	2.0625	21.74	201,800	13.316	22.04	204,600	13.686

TABLE 1.—(*Continued*)

Web thickness (in.)	Length, g (in.)	(*A*) Series A–B–H–K ($r_{A-B} = \frac{3}{4}$ in. and $r_{H-K} = \frac{1}{2}$ in.)			(*B*) Series HH–KK ($r = \frac{1}{2}$ in.)		
		Distance, y (in.)	Moment, M (in.-lb)	$\frac{1}{Z \times 10^8}$	Distance, y (in.)	Moment, M (in.-lb)	$\frac{1}{Z \times 10^8}$
(1)	(2)	(3)	(4)	(5)	(3)	(4)	(5)
(*g*) Connection 9: $h = 26.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$							
....	....	$y = 0.924\ h$			$y = 0.937\ h$		
$\frac{7}{8}$	1.8750	24.45	277,900	24.528	24.80	281,900	25.236
$\frac{15}{16}$	1.9375	24.48	270,500	22.554	24.82	274,200	23.184
1	2.0000	24.49	263,200	20.704	24.83	266,900	21.283
$1\frac{1}{16}$	2.0625	24.52	256,700	19.102	24.86	260,200	19.635
(*h*) Connection 10: $h = 29.5$; $t = \frac{7}{16}$; and $g_1 = 1.8125$							
....	....	$y = 0.923\ h$			$y = 0.936\ h$		
1	1.8125	27.18	354,600	37.058	27.57	359,600	38.129
$\frac{7}{8}$	1.8750	27.20	344,200	33.793	27.60	349,300	34.784
$\frac{15}{16}$	1.9375	27.22	337,700	31.042	27.63	339,900	31.984
$1\frac{1}{16}$	2.0000	27.25	326,100	28.535	27.64	330,800	29.358

Constants for Standard Connections

Computed Values of $1/Z$.—Table 1 includes values of $1/Z$ for all the standard AISC connections[8,9] with the exception of numbers 1 and 2 which, for obvious reasons, have been omitted. (In Cols. 5, Table 1, values are multiplied by 10^8 to obtain $1/Z$.) They were computed by Eq. 22.

The standard distance, $5\frac{1}{2}$ in.,[8,9] between the two inner rows of column rivets gives rise to a different value of g for each web thickness straddled by the connection. Therefore, since the connection constants are affected by any change in g, values for M and $1/Z$ for the complete range of web thicknesses likely to be encountered have been included in Table 1. Both the web thicknesses and the corresponding values of g are recorded. Also recorded are the web-connected leg, g_1; the height of the connection, h; the thickness of the connection angle, t; and the distance from the top of the connection to its neutral axis, y.

The tabulated values for M were computed by Eq. 31. The strengthening effect of the fillet has thus been taken into consideration and, therefore, these values for M may be used in the proposed new method of bracket design.

Comparison with Published Laboratory Results

Computed Slopes for Moment-Rotation Curves.—Table 2 contains the data necessary for computing the slopes of the moment-rotation curves for specimens tested by Mr. Rathbun,[3] as well as the corresponding slopes as computed by Eq. 22. It will be noticed that none of the four specimens fall among the AISC standard connections. Mr. Rathbun's moment-rotation curves for

these specimens[3] have been reproduced in Fig. 4 and, in order to establish a check between his laboratory measured results and those computed by Eq. 22, the latter have been plotted on the reproduced curves.

Attention is called to the consistent agreement between the computed slopes and the curves for specimens 5, 6, and 7. That the computed slope for

TABLE 2.—DATA FROM PUBLISHED LABORATORY RESULTS
($t = \frac{3}{8}$ IN. AND $g_1 = 1.8750$ IN.)

Specimen[a]	h	g	b	n	y	$1/(Z \times 10^3)$
4	9	2.1875	4	20.59	8.43	0.407
5	9	1.9375	6	18.52	8.51	0.577
6	15	2.1250	4	20.07	14.04	2.037
7	15	1.8750	6	18.00	14.17	2.906

[a] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, p. 527, Fig. 1.

specimen 4 is slightly too steep for the curve may be explained by the fact that specimen 4 is the weakest of those checked and that, therefore, the factors mentioned as having been neglected in the derivation of Eq. 22 might be expected to assume the greatest relative importance with respect to this specimen.

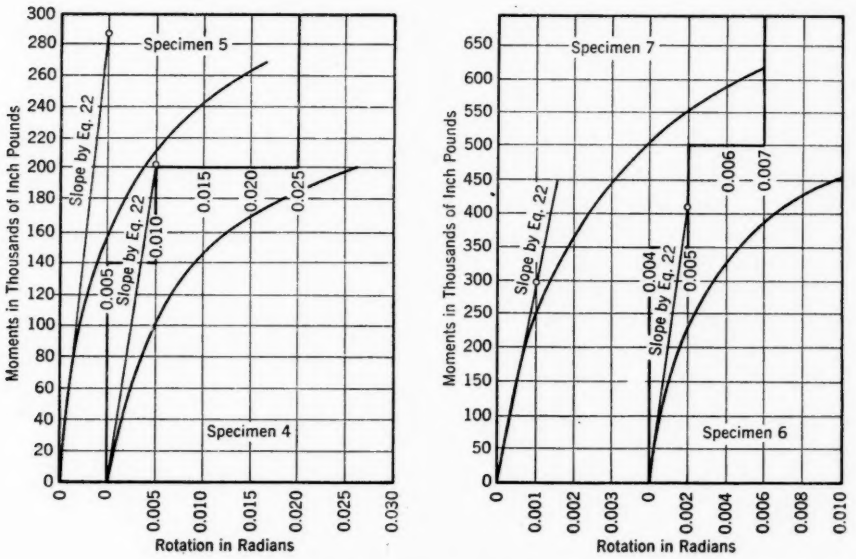

FIG. 4.—COMPUTED SLOPES PLOTTED ON MOMENT-ROTATION CURVES

In the case of specimen 7, on the other hand, the computed slope is barely steep enough. This is the strongest of the specimens checked and, in this instance, inclusion of the fillet effects in the computed slope might be expected to give it the necessary additional steepness.

Application of Eq. 31 to Bracket Design

It is quite generally recognized by the engineering profession that the present method of basing the design of certain types of steel brackets on the rivet strength often results in an overstress in the connection angles. Furthermore, the 1946 AISC specification[8,9] for permissible tension in rivets will heighten this tendency. Eq. 31 provides an alternate method of design for the the type of bracket indicated in Fig. 5, and one which is based on the critical bending stress in the angles.

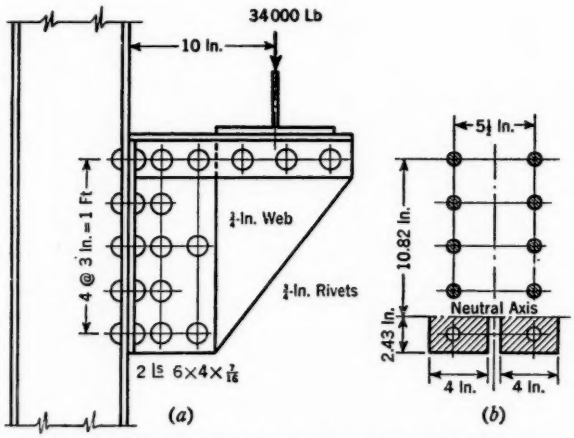

Fig. 5.—Bracket Loaded to Full Rivet Strength

Bracket Design Based on Rivet Strength.—In the following example the comparison to be demonstrated is that between the strength of the rivets connecting the column and the bending strength of the column-connected legs of the connection angles. The other rivets are a separate consideration and will not be treated in this example.

Although a direct approach (outside the scope of this paper) is possible, the usual procedure is to assume a bracket and then to test the sufficiency of the rivets in tension and shear, respectively, by the following equations:

$$f = \frac{M\,d}{2\,I} + \sqrt{\left(\frac{M\,d}{2\,I}\right)^2 + \left(\frac{P}{N\,a}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

and

$$v = \sqrt{\left(\frac{M\,d}{2\,I}\right)^2 + \left(\frac{P}{N\,a}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

in which f is the maximum tension in the topmost rivet, a distance of d inches from the neutral axis, due to combined bending and shear; and v is the corresponding maximum shear in the same rivet. These stresses are limited to 20,000 lb per sq in. and 15,000 lb per sq in., respectively. Symbol M denotes the moment from the eccentric load, P; N is the number of rivets through the column flange; a is the sectional area of one rivet shank; and I is the moment of inertia of the contact section (Fig. 5(*b*)) between the bracket and the column.

The shape of the contact section is found by taking static moments about the neutral axis as demonstrated by Thomas C. Shedd,[10] M. ASCE. For the bracket in Fig. 5, I is 217.8 in.^{4}; the distance, d, from the neutral axis to the topmost rivets is 10.82 in.; M is 340,000 in-lb; and N is assumed as 10.

Applying Eqs. 32 and 33, f and v are found to be 19,880 lb per sq in. and 11,430 lb per sq in., respectively, both of which are within the allowable limits specified by the AISC.[9,11] The rivets through the web of the bracket may be investigated by methods recommended by Mr. Shedd.[12]

Bracket Design Based on Angle Strength.—As may be observed from Table 1, the ratio between y and h is constant for a wide range of standard connections. For the series that includes connection K5, $y = 0.924\ h$. If this substitution for y be made in Eq. 18 and, if the fillet factor, 1.13, be applied,

$$M = \frac{h^2\ s\ t^2}{8.6\ g} \times \frac{4\ g + g_1}{2\ g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

It is evident from Eq. 34 that the resisting moment of the bracket, as governed by the connection angles, increases with the square of its height, as well as with the square of the thickness of the angles. Three direct methods of approach are possible for the design of a bracket for a given moment:

1. Select a standard connection from Table 1(A) or Table 1(B) which furnishes a sufficiently large resisting moment to satisfy the applied moment. The required thickness of web will be indicated at the same time; although, it should be reviewed for shear, buckling, etc.

2. Having decided upon a desirable size of connection angles and web thickness, compute the necessary height, h, or length of connection angles, by Eq. 34 or Eq. 35.

Eqs. 20 and 21 indicate upon what elements of the connection the ratio y/h depends. With that information at hand, the proper ratio may be found in Table 1. However, if the angles chosen are not listed in Table 1, resort must be had to a general form of Eq. 34 as follows:

$$M = \frac{h^2\ s\ t^2}{K\ g} \times \frac{4\ g + g_1}{2\ g + g_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

in which $K = 9/\ (y/h \times 1.13)$.

3. Having decided upon a desirable height of bracket and having made a tentative selection of the angles, as regards the length of their legs, the required thickness of the chosen angles may be computed by Eq. 35.

Applying the first method to the present example, Table 1(A) shows a resisting moment of only 80,900 in-lb for the assumed standard connection, K5. In fact, Table 1(A) indicates that, if a AISC standard connection is to be selected to resist the applied moment of 340,000 in-lb, it will require a number 10.

On the other hand, if it seems desirable to retain the assumed height of the bracket, the necessary thickness of angles may be found, in accordance with

[10] "Structural Design in Steel," by Thomas C. Shedd, John Wiley & Sons, Inc., New York, N. Y., 1934, p. 293.

[11] "Steel Construction," AISC Manual, New York, N. Y., 1947, p. 286.

[12] "Structural Design in Steel," by Thomas C. Shedd, John Wiley & Sons, Inc., New York, N. Y., 1934, p. 305.

method 3, by Eq. 34. A shorter method is as follows: The strength of the connection, as indicated by Eqs. 34 and 35, increases as the square of the thickness, therefore, $\left(\frac{t}{0.4375}\right)^2 = \frac{340{,}000}{80{,}900}$; and $t = 0.897$ in.

Concluding Remarks

Eqs. 18 through 35 apply only to the type of connection indicated in Fig. 3(*a*); that is, any connection that is made by straddling the web of the beam or the bracket with a pair of angles, the outstanding legs of which make the connection to the column, and which are not accompanied by seat angles or clip angles. However, the same reasoning that was employed in the derivation of the aforementioned equations may be applied to the split beam type of connection, to the seat and clip angle connections, or to a combination of the latter type and the present type under consideration. In the last-named combination type of connection, the equations become rather long and tedious; but they certainly provide an improvement over the present method of laboratory tests.

One conclusion that might be drawn from the foregoing example is that the present design practice tends toward the use of weak angles in connections. It is true that, in many cases, this is deliberate, in an attempt to approach simple beam conditions. On the other hand, economy might be achieved by stiffening these connections and by taking their restraint into consideration in design. New specifications[8,9] permit additional tension in rivets. Undoubtedly, this has been brought about by the published results of research on the tensile strength of rivets.[4,5,6] On the other hand, the thicknesses of standard connection angles still remain at $\frac{3}{8}$ in. and $\frac{7}{16}$ in. Examples of angle failure in which one or both legs of the angle have been strained out of shape, with no accompanying evidence of strain in the rivets, are common.

Certain factors that contribute to the angle of strain, ϕ, have been neglected in the foregoing analysis. Within the range of permissible working loads, however, the strain in the column-connected legs of the connection angles is so great by comparison that all other factors may be considered negligible. In the interest of simplicity, in fact, the same relative assumptions have been made as is the practice in computing secondary stresses in trusses.

APPENDIX. NOTATION

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Structural Analysis (ASA—Z10.8—1949), prepared by a committee of the American Standards Association, with Society representation, and approved by the Association in 1949. Subscripts *ab*, *bc*, etc., denote "end A of span AB," "end B of span BC," etc. The subscripts $A-B$, $H-K$, HH, etc., denote "of the standard connection series,[8,9] A, B, H, K, HH," etc. Symbol A denotes the fixed end of the column-connected leg of a connection angle, assumed to be at the inner rivet when there is more than one row of rivets; B is the heel end of either angle leg g or g_1; and C, the

fixed end of the web-connected leg of a connection angle assumed to be at the rivet nearest the end of the beam, also, with numerical subscripts, represents constants in the general form of the slope deflection equation.

a = area of a section taken across the rivet shank; also, used as a subscript in referring to the A-end of a beam;

b = the total length of the column-connected leg of a connection angle which is used to indicate the width of the compression area between the back of the angle and the face of the column; also, b is used as a subscript in referring to the B-end of a beam;

c = a subscript used in referring to the C-end of a beam;

d = the distance from the neutral axis of a group of rivets to any given rivet of the group;

E = modulus of elasticity;

f = maximum tensile stress in a rivet due to combined shear and tension;

g = the length of the column-connected leg of the angle, measured from the center of rivet A to the near face of the web-connected leg, as shown in Fig. 1;

g_1 = the length of the web-connected leg of the angle, measured from the center of rivet C to the near face of the column-connected leg, as shown in Fig. 1;

h = the total height of the connection;

I = moment of inertia; I_o = moment of inertia of some reference section, in this case, that of the prismatic portion of the angle leg;

K = a general constant term with suitable subscripts;

$k = g_1/(4\,g + 2\,g_1)$ or corresponding term for fillet-based equations;

M = with subscript, the moment at the end of an angle leg; without subscript, the resisting moment of the connection;

N = the number of rivets connecting the angles to the column;

$n = 1/k$, the ratio of the intensity of the compressive stress below the neutral axis to the tensile stress due to bending in the outstanding leg of the angle above the neutral axis;

P = a concentrated load;

r = without subscript, the radius of the angle fillet; with suitable subscript, the deflection of the heel of the angle divided by the effective length, g, of the leg;

s = without subscript, the allowable bending stress in the legs of the connection; with suitable subscript, the stress at the indicated point;

t = the nominal thickness of the angle;

v = maximum shearing stress in a rivet due to combined shear and tension;

y = the distance from the top of the angle to the neutral axis as governed by bending in the column-connected leg of the angle;

Z = a constant such that $Z = \phi/M$ and such that $1/Z$ equals the slope of the moment-rotation curve for the connection;

Δ_b = deflection of the heel of the angle at the top of the connection;

θ = the angle of rotation of the joint, as a whole, with appropriate subscript to indicate the rotating joint; and

ϕ = the angle of strain.

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